Prove that for any $a, b, c \geq 0$,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

This inequality is one of a suite of inequalities known collectively as the **Arithmetic–Geometric Mean Inequality**, or **AGM** inequality for short. (The arithmetic mean is the familiar "mean" or average on the left of the inequality, and the geometric mean is what we call the expression on the right.) That inequality says that for any set of non–negative numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{1}{n}(a_1 + a_2 + \ldots + a_n) \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

Okay, so now that we know what it's called, how do we go about proving it for this $n = 3$ case? As is typical of many "name brand" theorems, there are a number of different ways to proceed. The one presented here is the most basic in the sense that it uses only a little bit of algebra and a little bit of keeping track of what you already know.

The procedure goes as follows: first we prove the AGM is true for the $n = 2$ and $n = 4$ cases. These are not particularly hard and with them in hand, the $n = 3$ case is then very easy. The $n = 2$ case itself is easy. Letting a and b be our non-negative numbers we have

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

since any square is positive. Algebra does the rest. Multiplying out we get:

$$a - 2\sqrt{ab} + b \geq 0$$

or

$$\frac{a+b}{2} \geq \sqrt{ab}$$

With the $n = 2$ case done, the $n = 4$ case is also easy. Letting a, b, c and d be our non-negative numbers we have

$$\begin{aligned}
\frac{a+b+c+d}{4} &= \frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right) \\
&\geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)} \\
&\geq \sqrt{\sqrt{ab}\sqrt{cd}} \\
&= \sqrt[4]{abcd}
\end{aligned}$$

where for the each inequality sign I've used the $n = 2$ AGM inequality. Now we can go back and do the $n = 3$ case pretty quickly. Notice that

$$\begin{aligned}
\frac{a+b+c}{3} &= \frac{1}{4}\left(a + b + c + \frac{a+b+c}{3}\right) \\
&\geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}
\end{aligned}$$

where for the inequality we used the $n = 4$ AGM. But the above equation can be written

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{4}} \left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

or

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

And now simply raising each side of the above equation to the $\frac{4}{3}$ power finishes us off.

Note: Just to give a flavor of the many different ways one can prove our problem, we present here two more proofs that use more "advanced" mathematics. Note that they are both much shorter.

Alternate Proof 1 We can consider this problem from an optimization viewpoint by restating it as follows: "What is the maximum value that $f(a,b,c) = \sqrt[3]{abc}$ can achieve if we must have in addition $\frac{a+b+c}{3} = A$ for some fixed A?" (A stands for average.) This is a simple Lagrange multiplier problem from calculus which involves finding the critical values for the function

$$g(a,b,c,\lambda) = \sqrt[3]{abc} - \lambda \left(\frac{a+b+c}{3} - A\right)$$

Taking derivatives as usual we arrive at the conditions

$$(bc)^{1/3}a^{-\frac{2}{3}} = (ac)^{1/3}b^{-\frac{2}{3}} = (ab)^{1/3}c^{-\frac{2}{3}}$$
$$\tfrac{a+b+c}{3} = A$$

Simplifying these conditions we arrive at

$$a = b = c$$
$$\tfrac{a+b+c}{3} = A$$

or $a = b = c = A$. It is easy to see that this is a maximum, and thus the maximum value of $f(a,b,c)$ is at $f(A,A,A) = A$. Restating that previous sentence as mathematics says that

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}$$

as desired.

Alternate Proof 2 A function $f(x)$ is called *convex* if its second derivative is always positive. One of the desirable properties that convex functions have is that, for any set of positive numbers λ_i we have that add up to one, the function satisfies

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

Why is this useful? Well, $f(x) = -\ln x$ is certainly convex, and $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$. Using the convexity property now gives us

$$-\frac{1}{3}\ln a - \frac{1}{3}\ln b - \frac{1}{3}\ln c \geq -\ln\left(\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c\right)$$

Properties of ln make this

$$-\ln(abc)^{1/3} \geq -\ln\left(\frac{a+b+c}{3}\right)$$

and now taking negative of both sides (which reverses the inequality) and then taking the exponential of both sides gives us our result.