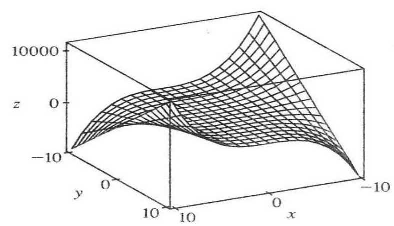
Section 11.7 - 6,8,30,36,46

6, 8. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

6. $f(x,y) = x^3y + 12x^2 - 8y$

Solution:

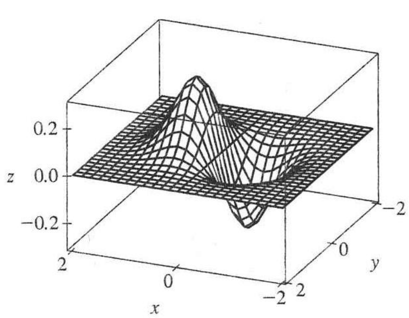
$f(x,y) = x^3y + 12x^2 - 8y \Rightarrow f_x = 3x^2y + 24x$, $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \Rightarrow y = -4$. Thus, the only critical point is $(2,-4)$. $D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.



8. $f(x,y) = xe^{-2x^2-2y^2}$

Solution:

$f(x,y) = xe^{-2x^2-2y^2} \Rightarrow f_x = (1-4x^2)e^{-2x^2-2y^2}$, $f_y = -4xye^{-2x^2-2y^2}$, $f_{xx} = (16x^2-12)e^{-2x^2-2y^2}$, $f_{xy} = (16x^2-4)ye^{-2x^2-2y^2}$, $f_{yy} = (16y^2-4)xe^{-2x^2-2y^2}$. Then $f_x = 0$ implies $1 - 4x^2 = 0 \Rightarrow x = \pm\frac{1}{2}$, and substitution into $f_y = 0 \Rightarrow -4xy = 0$ gives $y = 0$, so the critical points are $\left(\pm\frac{1}{2}, 0\right)$. Now $D\left(\frac{1}{2},0\right) = (-4e^{-1/2})(-2e^{-1/2}) - 0^2 = 8e^{-1} > 0$ and $f_{xx}\left(\frac{1}{2},0\right) = -4e^{-1/2} < 0$, so $f\left(\frac{1}{2},0\right) = \frac{1}{2}e^{-1/2}$ is a local maximum. $D\left(-\frac{1}{2},0\right) = (4e^{-1/2})(2e^{-1/2}) - 0^2 = 8e^{-1} > 0$ and $f_{xx}\left(-\frac{1}{2},0\right) = 4e^{-1/2} > 0$, so $f\left(-\frac{1}{2},0\right) = \frac{1}{2}e^{-1/2}$ is a local minimum.

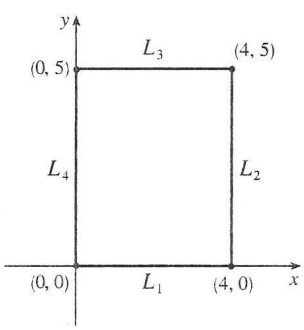


30. Find the absolute maximum and minimum values of f on the set D.

$$f(x, y) = 4x + 6y - x^2 - y^2,$$
$$D = \{(x, y) | 0 \le x \le 4, 0 \le y \le 5\}$$

Solution:

$f_x(x, y) = 4-2x$ and $f_y(x, y) = 6-2y$, so the only critical point is $(2, 3)$ (which is in D) where $f(2, 3) = 13$. Along $L_1 : y = 0$, so $f(x, 0) = 4x - x^2 = -(x-2)^2 + 4$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2, 0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0, 0) = f(4, 0) = 0$. Along $L_2 : x = 4$, so $f(4, y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(4, 3) = 9$ and a minimum value when $y = 0$ where $f(4, 0) = 0$. Along $L_3 : y = 5$, so $f(x, 5) = -x^2 + 4x + 5 = -(x-2)^2 + 9$, $0 \le x \le 4$, which has a maximum value when $x = 2$ where $f(2, 5) = 9$ and a minimum value both when $x = 0$ and $x = 4$ where $f(0, 5) = f(4, 5) = 5$. Along $L_4 : x = 0$, so $f(0, y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(0, 3) = 9$ and a minimum value when $y = 0$ where $f(0, 0) = 0$. Thus the absolute maximum is $f(2, 3) = 13$ and the absolute minimum is attained at both $(0, 0)$ and $(4, 0)$, where $f(0, 0) = f(4, 0) = 0$.



36. Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Solution:

Here the distance d from a point on the plane to the point $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$, where $z = 4 - x + y$. We can minimize $d^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (1-x+y)^2$, so $f_x(x, y) = 2(x-1)+2(1-x+y)(-1) = 4x-2y-4$ and $f_y = 2(y-2)+2(1-x+y) = 4y-2x-2$. Solving $4x - 2y - 4 = 0$ and $4y - 2x - 2 = 0$ simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $\left(\frac{5}{3}, \frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1, 2, 3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

46. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

Solution:

The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy+2V(x^{-1}+y^{-1})$. Then $C_x = 5y-2Vx^{-2}$, $C_y = 5x-2Vy^{-2}$, $f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

Section 11.8 - 6,8,18,28

6, 8. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

6. $f(x, y) = e^{xy}$; $x^3 + y^3 = 16$

Solution:

$f(x, y) = e^{xy}$, $g(x, y) = x^3 + y^3 = 16$, and $\nabla f = \lambda \nabla g \Rightarrow < ye^{xy}, xe^{xy} >=< 3\lambda x^2, 3\lambda y^2 >$, so $ye^{xy} = 3\lambda x^2$ and $xe^{xy} = 3\lambda y^2$. Note that $x = 0 \Leftrightarrow y = 0$ which contradicts $x^3 + y^3 = 16$, so we may assume $x \neq 0$, $y \neq 0$, and then $\lambda = ye^{xy}/(3x^2) = xe^{xy}/(3y^2) \Rightarrow x^3 = y^3 \Rightarrow x = y$. But $x^3 + y^3 = 16$, so $2x^3 = 16 \Rightarrow x = 2 = y$. Here there is no minimum value, since we can choose points satisfying the constraint $x^3 + y^3 = 16$ that make $f(x, y) = e^{xy}$ arbitrarily close to 0 (but never equal to 0). The maximum value is $f(2, 2) = e^4$.

8. $f(x, y, z) = 8x - 4z$; $x^2 + 10y^2 + z^2 = 5$

Solution:

$f(x, y, z) = 8x - 4z$, $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f =< 8, 0, -4 >$, $\lambda \nabla g = < 2\lambda x, 20\lambda y, 2\lambda z >$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \frac{4}{\lambda}$, $y = 0$, and $z = -\frac{2}{\lambda}$. But $5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10(0)^2 + \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{20}{\lambda^2} \Rightarrow \lambda = \pm 2$, so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$ and the minimum is $f(-2, 0, 1) = -20$.

18. Find the extreme values of f on the region described by the inequality.

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5,\ x^2 + y^2 \leq 16$$

Solution:

$f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f =< 4x - 4, 6y >=< 0, 0 > \Rightarrow x = 1,\ y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x, y) = x^2 + y^2 = 16 \Rightarrow \lambda\nabla g =< 2\lambda x, 2\lambda y >$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. Thus, the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.

28. Use Lagrange multipliers to give an alternative solution to the indicated exercise in Section 11.7.

Exercise 36 - Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Solution:

Let $f(x, y, z) = d^2 = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$, then we want to minimize f subject to the constraint $g(x, y, z) = x - y + z = 4$. $\nabla f = \lambda \nabla g \Rightarrow$
$< 2(x-1), 2(y-1), 2(z-3) >= \lambda < 1, -1, 1 >$, so $x = (\lambda+2)/2$, $y = (4-\lambda)/2$, $z = (\lambda+6)/2$. Substituting into the constraint equation gives $\frac{\lambda+2}{2} - \frac{4-\lambda}{2} + \frac{\lambda+6}{2} = 4 \Rightarrow \lambda = \frac{4}{3}$, so $x = \frac{5}{3}$, $y = \frac{4}{3}$, and $z = \frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1, 2, 3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.