Section 9.3: 2,10,18,22,24,32,38,49,50

2, 10. Find $\mathbf{a} \cdot \mathbf{b}$

2. $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{6}$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is 45^o

Solution:

By the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = (3)(\sqrt{6}) \cos 45^o = 3\sqrt{6}\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2} \cdot 2\sqrt{3} = 3\sqrt{3}$.

10. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

Solution:

$\mathbf{a} \cdot \mathbf{b} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}) = (3)(4) + (2)(0) + (-1)(5) = 7$.

18. Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k},\ \mathbf{b} = 4\mathbf{i} - 3\mathbf{k}$$

Solution:

$|\mathbf{a}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$, $|\mathbf{b}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5$, and $\mathbf{a} \cdot \mathbf{b} = (1)(4) + (2)(0) + (-2)(-3) = 10$. Then $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|a||b|} = \frac{10}{3 \cdot 5} = \frac{2}{3}$ and $\theta = \arccos\frac{2}{3} \approx 48^o$

22. Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\mathbf{u} =< -3, 9, 6 >$, $\mathbf{v} =< 4, -12, -8 >$

(b) $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

(c) $\mathbf{u} =< a, b, c >$, $\mathbf{v} =< -b, a, 0 >$

Solution:

(a) Because $\mathbf{u} = -\frac{3}{4}\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors (and thus are not orthogonal).

(b) $\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-1)(-1) + (2)(1) = 5 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Also, $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

(c) $\mathbf{u} \cdot \mathbf{v} = (a)(-b) + (b)(a) + (c)(0) = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (and not parallel).

24. For what values of b are the vectors $< -6, b, 2 >$ and $< b, b^2, b >$ orthogonal?

Solution:

$< -6, b, 2 >$ and $< b, b^2, b >$ are orthogonal when $< -6, b, 2 > \cdot < b, b^2, b >= 0 \Leftrightarrow (-6)(b) + (b)(b^2) + (2)(b) = 0 \Leftrightarrow b^3 - 4b = 0 \Leftrightarrow b(b+2)(b-2) = 0 \Leftrightarrow b = 0$ or $b = \pm 2$.

32. Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Solution:

$|\mathbf{a}| = \sqrt{1+1+1} = \sqrt{3}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{1-1+1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{1}{\sqrt{3}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}} \cdot \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} = \frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

38. A tow truck drags a stalled car along a road. The chain makes an angle of 30^o with the road and the tension in the chain is 1500N. How much work is done by the truck in pulling the car 1km?

Solution:

Here $|\mathbf{D}| = 1000$m, $|\mathbf{F}| = 1500$N, and $\theta = 30^o$. Thus $W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta = 1500(1000)(\frac{\sqrt{3}}{2}) = 750,000\sqrt{3}$ joules.

49. Prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

Solution:

$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta|$. Since $|\cos\theta| \leq 1$, $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$.
Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

50. The Triangle Inequality for vectors is

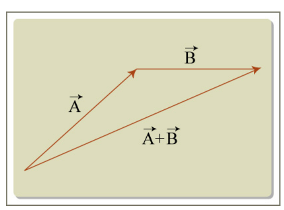
$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.

(b) Use the Cauchy-Schwarz Inequality from Exercise 49 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

Solution:

(a) The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.



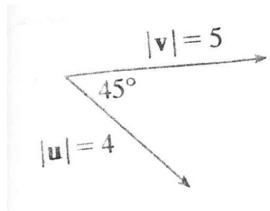
(b)

$$\begin{aligned}
|\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a}) + 2(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \qquad \text{[by the Cauchy-Schwarz Inequality]} \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2
\end{aligned}$$

Thus, taking the square root of both sides, $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

Section 9.4: 2,5,8,22,24

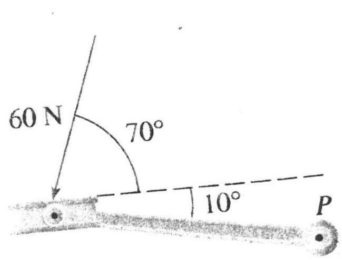
2. Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or our of the page.



Solution:

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta = (4)(5)\sin 45^o = 20 \cdot \frac{\sqrt{2}}{2} = 10\sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

5. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18cm long. Find the magnitude ofthe torque about P.



Solution:

The magnitude of the torque is $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta = (0.18\text{m})(60\text{N}) \sin(70 + 10)^o = 10.8 \sin 80^o \approx 10.6\text{N} \cdot \text{m}$.

8. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} =< 1, 1, -1 >, \ \mathbf{b} =< 2, 4, 6 >$$

Solution:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \mathbf{k} \\ &= [6 - (-4)]\mathbf{i} - [6 - (-2)]\mathbf{j} + (4 - 2)\mathbf{k} = 10\mathbf{i} - 8\mathbf{j} + 2\mathbf{k} \end{aligned}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} =< 10, -8, 2 > \cdot < 1, 1, -1 >= 10 - 8 - 2 = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} =< 10, -8, 2 > \cdot < 2, 4, 6 >= 20 - 32 + 12 = 0$, so $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

22. Find the area of the parallelogram with vertices $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

Solution:

The parallelogram is determined by the vectors $\vec{KL} =< 0, 1, 3 >$ and $\vec{KN} =< 2, 5, 0 >$, so the area of the parallelogram $KLMN$ is

$$|\vec{KL} \times \vec{KN}| = \left\| \begin{matrix} i & j & k \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{matrix} \right\| = |(-15)\mathbf{i} - (-6)\mathbf{j} + (-2)\mathbf{k}| = |-15\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}| = \sqrt{265} \approx 16.28$$

24. (a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

$$P(-1, 3, 1), \ Q(0, 5, 2), \ R(4, 3, -1)$$

Solution:

(a) $\vec{PQ} =< 1, 2, 1 >$ and $\vec{PR} =< 5, 0, -2 >$, so a vector orthogonal to the plane through P, Q, and R is $\vec{PQ} \times \vec{PR} =< (2)(-2) - (1)(0), (1)(5) - (1)(-2), (1)(0) - (2)(5) >$ $=< -4, 7, -10 >$ [or any scalar multiple thereof].

(b) The area of the parallelogram determined by $\vec{PQ}$ and $\vec{PR}$ is $|\vec{PQ} \times \vec{PR}| = |< -4, 7, -10 >| = \sqrt{16 + 79 + 100} = \sqrt{165}$, so the area of the triangle PQR is $\frac{1}{2}\sqrt{165}$.