General Examination: Real Variables

The answers without reasonable supporting work will not receive credit.

Problem 1. Evaluate the limit and justify the result:

$$\lim_{n\to\infty} \quad n \int_0^1 \frac{x^{n-1}}{2020+x} dx$$

Problem 2. Consider the identity operator $I(x) = x$.

(a) Let $I:\ L^1[0,1] \mapsto L^2[0,1]$. Is operator I bounded?
Prove or find a counter example: If $\|x_n\|_1 \to 0$ as $n \to \infty$, then $\|x_n\|_2 \to 0$.

(b) Let $I:\ L^2[0,1] \mapsto L^1[0,1]$. Is operator I bounded?
Prove or find a counter example: If $\|x_n\|_2 \to 0$ as $n \to \infty$, then $\|x_n\|_1 \to 0$.

Problem 3. Let X be a Banach space and suppose that mapping $f:\ X \mapsto X$ is contractive. Define $g(x) = x - f(x)$, $x \in X$. Show that the mapping $g:\ X \mapsto X$ is one-to-one and onto. Also show that g and its inverse are continuous.

Problem 4. Consider the sequence of functions $\{\cos(nr)\}_{n=1}^{\infty}$ defined on the set $\mathbb{Q}$ of rational numbers, $r \in \mathbb{Q}$. Prove that there is a subsequence of $\{\cos(nr)\}$ that converges pointwise on $\mathbb{Q}$ to a real-valued function.

Problem 5. Consider the differential equation

$$x'(t) = \cos\left(x^2 + 3x + \frac{\pi}{2}\right), \quad 0 \le t \le T < \infty,$$

with initial condition $x(0) = x_0$ and solutions in $C[0,T]$.
Let $K \subset C[0,1]$ be the closure in space $C[0,T]$ of all solutions $x(t)$ to the above initial value problem for all x_0 such that $|x_0| < 1$. Let $y(t) = e^t$. Prove that there exists function $z \in K$ such that

$$\|z - y\| = \mathrm{dist}(y, K).$$

Here $\mathrm{dist}(y,K)$ is the distance in space $C[0,T]$ between point y and the set K.

Problem 6. Let us consider Sobolev space $W^{2,1}[0,1]$ of functions, which are in $L^2[0,1]$ along with their derivatives. Sobolev space is equipped with the following inner product and is complete:

$$(x,y) = \int_0^1 [x(t)y(t) + x'(t)y'(t)]dt.$$

(a) Prove that $W^{2,1}[0,1] \subset C[0,1]$;

(b) Let $f(x) = x(0)$. Is it possible to find $y \in W^{2,1}[0,1]$ such that $\forall x \in W^{2,1}[0,1],\ f(x) = (x,y)$? If yes, find such function $y(t)$.