Doctoral Program | **Department of Mathematical Sciences**
Spring 2019 | Stevens Institute of Technology

General Examination: Real Variables

Problem 1.

(a) State Baire Category Theorem.

(b) Prove that in an infinite-dimensional Banach space X, there is no countable set B such that every $x \in X$ is a linear combination of a finite number of elements of B.

Problem 2. Let a sequence (f_n) be uniformly bounded almost everywhere and weakly convergent to zero in $L^1(0,1)$. Does $f_n \to 0$ in norm in space $L^1(0,1)$?

Problem 3. Let (X,d) be a compact metric space, and let $T : X \to X$ be such that $d(T(x),T(y)) < d(x,y)$ for all $x,y \in X$, $x \neq y$. Prove that

(a) T has a unique fixed point in M;

(b) the iterations $x_{n+1} = T(x_n)$ converge to the fixed point for any starting point in T.

Problem 4. Does there exist a sequence $x \in l^p$, $p > 1$, such that $x \notin l^q$ for all $1 \leq q < p$?

Problem 5. Let $E \subseteq \mathbb{R}^n$. Let m^* denote the Lebesgue outer measure.

(a) Is it true that if $m^*(E) = 0$ then E has empty interior?

(b) Is it true that if E has empty interior then $m^*(E) = 0$?

Problem 6. Find the limit and justify your conclusion:

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x^n)}{x^n} dx.$$