General Examination: Algebra

Solve at least three items from each problem.
(Items are labeled by (a), (b), (c) and (d).)

Problem 1.

(a) Let $S = \mathbb{R}\backslash\{1\}$. Define a binary operation $*$ on S by $a * b = a + b + ab$.
   (i) Prove that $(S, *)$ is an abelian group.
   (ii) Find all elements of order 3 in $(S, *)$.

(b) Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $U(n)$ the group of units of $\mathbb{Z}_n$. Show that if $n > 2$ then $U(n)$ has an element of order 2.

(c) Find an element of largest possible order in the group of permutations S_{10}.

(d) Denote by D_n the nth dihedral group (the group of rigid motions of a regular n-gon). Find the center of the group D_8. [Recall that the center $Z(G)$ of a group G consists of all elements $g \in G$ such that $gx = xg$ for any element $x \in G$]

Problem 2.

(a) Let H be a subgroup of index 2 of a group G. Show that H is a normal subgroup of G and find the factor-group G/H.

(b) Let $T = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ be a set of 2x2 matrices over $\mathbb{R}$ with determinant 1.
   (i) Show that T is a group.
   (ii) Let U be a subset of T consisting of all matrices of the type $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $c \in \mathbb{R}$. Show that U is a normal subgroup of T.
   (iii) Show that U is isomorphic to the addtive group $\mathbb{R}^+$ and the factor-group T/U is isomorphic to the multiplicative group $\mathbb{R}^*$.

(c) Let J be an ideal in a ring R. Prove that

$$I = \{r \in R \mid rt = 0 \text{ for every } t \in J\}$$

is also an ideal in R.

(d) Show (using Fermat's Little Theorem) that if $p = 4n+3$ is prime then there is no solution to the equation $x^2 = -1(\text{mod } p)$.

Problem 3.

(a) Let $F = \mathbb{Z}_2[x]/(x^3 + x + 1)$. Show that F is a field and find $(x^2 + x + 1)^{-1}$ in F.

(b) Show that $\sqrt[2]{3} + \sqrt[3]{5}$ is algebraic over $\mathbb{Q}$ and find its minimal plynomial.

(c) Find the Galois group of the polynomial $x^4 + 2x^2 - 8$ in $\mathbb{Q}[x]$.

(d) Let F be a finite field of characteristic 2. Prove that every element in F is a square.