General Examination: Real Variables

Problem 1. In ℓ^∞ (that is, the space of bounded sequences with the norm $\|(x_1, x_2, \ldots)\|_\infty = \sup_{k\geq 1}|x_k|$), find a descending system of closed sets

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$$

that do not have a common point, i.e.,

$$\bigcap_{k=1}^{\infty} C_k = \emptyset.$$

Problem 2. Show that if f is an a.e. nonnegative Lebesgue integrable over $[a,b]$ function, then its harmonic average is not more than its arithmetic average, that is

$$\left(\frac{1}{b-a}\int_{[a,b]}\frac{1}{f}dx\right)^{-1} \leq \frac{1}{b-a}\int_{[a,b]} f dx.$$

For which functions is the equality achieved?

Problem 3. (a) Prove that any finite dimensional subspace of Banach spaces is closed.
(b) Evaluate the distance in $C[0,1]$ from $x = 1+t$ to the subspace $L = \text{Span}\{t^2\}$.

Problem 4. For $x \in \ell^1$, define

$$\|x\|' = 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)|x_n|.$$

(a) Show that $\|\cdot\|'$ is a norm on ℓ^1.

(b) Show that $\|\cdot\|'$ is equivalent to $\|\cdot\|_1$.

(c) Show that ℓ^1 is complete with respect to $\|\cdot\|'$.

Problem 5. Evaluate

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{e^{-x^2/n}}{1+x^2}dx.$$

Problem 6. Is there an *open* subset of $\mathbb{R}$ of Lebesgue measure at most $\frac{1}{2}$ such that its closure has Lebesgue measure 1?