Doctoral Program | **Department of Mathematical Sciences**
Spring 2018 | Stevens Institute of Technology

General Examination: Real Analysis

Throughout this paper, m denotes Lebesgue measure on $\mathbb{R}$.

Problem 1. Give an example of an open set $A \subseteq \mathbb{R}$ such that $m(A) < m(\overline{A})$, where $\overline{A}$ is the closure of the set A.

Problem 2. Let $\mathcal{P}$ be the subspace of $C[1,\infty)$ with uniform metric that consists of functions that are bounded by a function of the form $1/x^b$ ($b \in \mathbb{R}$, $b > 0$), i.e.,

$$\mathcal{P} = \{f \in C[1,\infty) \mid \exists b \in \mathbb{R}, b > 0 \text{ s.t. } |f(x)| \leq 1/x^b \text{ on } \mathbb{R}\}.$$

Is $\mathcal{P}$ complete?

Problem 3. Let $f \geq 0$ on $[0,1]$ be measurable. Suppose $\int_{[0,1]} f^n \, dm = C < \infty$ for all $n = 1,2,\ldots$. Prove that there is a measurable subset $B \subseteq [0,1]$ s.t. $f = \chi_B$ a.e. on $[0,1]$. (Here χ_B denotes the characteristic function of the set B.)

Problem 4. (a) Prove that

$$\int_0^{\pi/2} \sqrt{x \sin x} \, dx \leq \frac{\pi}{2\sqrt{2}}.$$

(b) Prove that the inequality is, in fact, strict.

Problem 5. Is linear operator $A : l^1 \mapsto l^1$,

$$Ax = \left(\left(1 - \frac{1}{3}\right) x_1, \left(1 - \frac{1}{3} + \frac{1}{9}\right) x_2, \ldots, \left(1 - \frac{1}{3} + \frac{1}{9} - \ldots + \frac{(-1)^n}{3^n}\right) x_n, \ldots \right)$$

bounded? If yes, evaluate its norm.

Problem 6. Is $\mathcal{F} = \{\tan(ax) \mid -\frac{1}{2} \leq a \leq 1\}$ pre-compact as a subspace of $C[-1,1]$ with the uniform metric?