General Examination

Probability and Stochastic Processes

Problem 1. For the potential loss X with probability density

$$f(x) = \frac{\alpha b^\alpha}{x^{\alpha+1}}, \qquad \text{for } x \geq b,\ \alpha > 0 \text{ and } d > b > 0,$$

consider one insurance policy with the following clauses:

- Deductible The insurer only takes care of the loss above $d > 0$ if the owner gets the loss X larger than d;
- Coverage limit The insurer only takes care of the loss below $l > d$ if the owner gets the loss X larger than l.
- Distorted distribution The insurer may base the premium of the loss X on the distorted version Z with survival function $h(S_X)$, where S_X is the survival function of X, and the continuous strictly increasing $h : [0,1] \mapsto [0,1]$ is called a distortion function.

(a) (5pts) Determine the probability distribution μ_X and the probability distribution function F_X of X.

(b) (5pts) Determine the induced distribution μ_Y and the probability distribution function F_Y of the potential attained loss Y of the insurer.

(c) (5pts) Compute the mathematical expectation of Y.

(d) (5pts) If there is one density of the probability distribution μ_Z with respect to μ_X? Derive it if yes.

(e) (5pts) Compute the mathematical expectation of Z, which is usually considered as the premium of the loss X for some suitable h.

Problem 2. For a rolling die, assume that the probability of each possible number of points X is proportional to $7 - X$.

(a) (5pts) Formulate the probability space for this random experiment.

(b) (5pts) Assume an insider always revealing the parity of X. Determine the sub-σ-field $\mathscr{G}$ due to the parity and the probability $P(X \leq 3 \mid \mathscr{G})$.

(c) (10pts) Determine the expectation $\mathrm{E}[X \mid \mathscr{G}]$.

(d) (5pts) Verify the duplicate expectation $\mathrm{E}[\mathrm{E}[X \mid \mathscr{G}]] = \mathrm{E}[X]$.

Problem 3. During each unit of time, (i) either 0 (with probability $1 - \lambda$) or 1 (with probability λ) customer arrives for service and joins a single line, and (ii) independently of new arrivals, a single service is completed with probability p or continues into the next period with probability $1 - p$. Let X_n be the total number of customers (waiting in line or being in service) at the nth unit of time and assume $0 < \lambda < p < 1$.

(a) (5pts) Show that $\{X_n\}$ is a birth-death chain on $S = \{0, 1, 2, \cdots\}$.

(b) (10pts) Discuss transience, recurrence, positive-recurrence of the chain.

(c) (5pts) Determine the invariant initial distribution π.

(d) (5pts) Evaluate $\mathrm{E}_\pi[X_n]$.

Problem 4. Let $\{N_t,\ t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and $\{Y_n,\ n \geq 1\}$ independent of $\{N_t,\ t \geq 0\}$ be i.i.d. jumps with

$$\mathrm{P}(Y_i = 1) = \mathrm{P}(Y_i = 2) = 1/2, \quad i = 1, 2, \cdots .$$

Denote $X_t = \sum_{n=0}^{N_t} Y_n$ for $t \geq 0$.

(a) (5pts) Evaluate the expectation $\mathrm{E}[X_t]$.

(b) (10pts) Show that the chain $\{X_t,\ t \geq 0\}$ has Markov property.

(c) (5pts) Determine the infinitesimal transition generator $\boldsymbol{Q}$.

(d) (5pts) Evaluate the distribution of the sojourn time at each state.