General Examination: Numerical Analysis

Problem 1. Neglecting round off error, how many terms of the McLaurin series for $\sin x$ are required to obtain a maximum error of less than 10^{-7},

(i) in the range $[0, \pi/2]$?

(ii) in the range $[0, \pi]$?

Provide a detailed verification of each of your answers.

Problem 2. Let $\{\phi_1, \phi_2, \ldots, \phi_k\}$ be a set of orthogonal basis functions (*discrete or continuous*) with respect to a real inner product and let $F(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_k\phi_k(x)$ be the least squares approximation to a known function $f(x)$ so that the mean squared error $E = \|f(x) - F(x)\|^2$ is minimized. Show the following:

(a) $\|F(x)\|^2 = \sum_{i=1}^{k} c_i^2 \|\phi_i(x)\|^2$, hence if $\phi_1, \phi_2, \ldots, \phi_k$ are normalized then $\|F(x)\|^2 = \sum_{i=1}^{k} c_i^2$.

(b) The minimized error is $E_{\min} = \|f(x)\|^2 - \|F(x)\|^2$.

(c) Use the Legendre polynomials

$$P_n(x) = \frac{1}{2^n\, n!}\left[\frac{d^n}{dx^n}\left(x^2 - 1\right)^n\right], \qquad n = 1, 2, \ldots$$

as basis functions to find the linear least squares polynomial approximation $F_k(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_k\phi_k(x)$ of degree k to $f(x) = e^x$ on $[-1, 1]$ for $k = 0, 1, 2, 3$. Then use the results of part (b) to calculate the mean squared error E_k for each approximation F_k; $k = 0, 1, 2, 3$.

Problem 3. Let A be a non-singular $n \times n$ matrix whose k^{th} column is X_k, $1 \le k \le n$.

(a) Determine an $n \times n$ lower triangular matrix L_k which differs from the identity I_n only in column k, such that

$$\text{If} \quad X_k = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ x_{k+1,k} \\ \vdots \\ x_{n-1,k} \\ x_{n,k} \end{pmatrix} \quad \text{then} \quad L_k X_k = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

State any assumptions that need to be made to justify the form of L_k.

(b) Provide an explicit form for L_k and show that L_k^{-1} has the same form as L_k.

(c) Let $L = L_1^{-1} L_2^{-1} \dots L_n^{-1}$. Show that $L^{-1}A = U$ is non-singular upper triangular, hence $A = LU$.

(d) Under the assumptions above that $A = LU$, describe how the factorization $A = LU$ is used to solve the linear system $AX = B$ where B is an $n \times 1$ vector.