General Examination: Complex Variables

Problem 1. Let $f(z)$ be analytic in the region $|z| > 1$. Suppose that $\lim_{z\to\infty} f(z) = 0$. Show that

$$\frac{1}{2\pi i}\int_{|\zeta|=2}\frac{f(\zeta)}{\zeta - z}d\zeta = -f(z)$$

whenever $|z| > 2$.

Problem 2. Let G be a connected open set on the complex plane, and let $z^* \in G$. Suppose a non-constant function f is analytic in $G\setminus\{z^*\}$. Show that if z^* is a limit of zeros of f, then it is an essential singularity of f.

Problem 3. Let $p_n(z) = z^n - z - 1$. Prove that for any given $r < 1 < R$, there is an N such that all zeros of $p_n(z)$ lie in the annulus $r \le |z| \le R$ for all $n \ge N$.