General Examination: Algebra

To pass the exam, solve at least two problems from each part and at least eight problems in total.

Group theory

Problem 1. Prove that a group G is finitely generated if and only if any increasing sequence $H_1 \leq H_2 \leq \dots$ of subgroups of G stabilizes, i.e. $H_i = H_j$ for $i, j \geq k$ starting from some k.

Problem 2. Compute the order of the automorphism group $\mathrm{Aut}(\mathbb{Z}_{p^k})$ of the cyclic group $\mathbb{Z}_{p^k}$ of a primary order p^k.

Problem 3. Let G be a group of order pq^2 where $p > q$ are prime numbers. Prove that G is solvable.

Problem 4. Let A be a finitely generated abelian group such that any subgroup of A is a direct summand of A. Prove that A is a finite cyclic group.

Ring theory and polynomials

Problem 5. Compute the number of all ideals of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

Problem 6. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

Problem 7. Let R be a commutative ring with identity. Let M and N be distinct maximal ideals of R. Prove that $R/(M \cap N)$ is isomorphic to the direct sum of two fields.

Problem 8. Let a and p be integers, p prime and $a \neq \pm(p+1)$. Prove that the polynomial $px^4 + ax + 1$ is irreducible over the field of rationals.

Field theory and Galois theory

Problem 9. Prove that the product of all non-zero elements of a finite field is -1.

Problem 10. Let H be an extension of a field F and $[H : F] = n$. Let $f \in F[x]$ be an irreducible polynomial of degree m such that $\gcd(m, n) = 1$. Prove that f has no root in H.

Problem 11. Let a field H be a finite dimensional extension of a field F and G the group of F-automorphisms of H. Prove that $|G| \leq [H : F]$.

Problem 12. Let $F = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{1 + \sqrt{2}}$. Find the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ up to isomorphism.