General Examination: Partial Differential Equations (PDEs)

Solve each problem on a separate sheet of paper.

Problem 1. An oscillating elastic string of length L has the left end fixed at $x = 0$ without the displacement with respect to the horizontal coordinate axis, i.e., $u(0,t) = 0$. The right end $x = L$ of the string is attached to a spring with elasticity k, the spring can move in the vertical direction only. One end of the spring is attached to the right end of the string whereas the other end of the spring is fixed at point (L,h). The unknown function $u(x,t)$ is the vertical deviation of the string from horizontal x-axis. The linear density of the string is ρ =const and the tension force is a known value T = const. The string undergoes a resistance force proportional to the deviation. Then, assuming that the u-axis is directed upward, the problem becomes as follows.

$$\rho u_{tt} = Tu_{xx} - ru, \quad u(0,t) = 0, \quad Tu_x(L,t) + k(u(L,t) - h) = 0.$$

(a) Until time $t = 0$ the string rests in an equilibrium position. Find this equilibrium position of the string.

(b) Assume that at time $t = 0$ the left end of the string starts to oscillate by law $u(0,t) = \sin t$. Find the solution to the boundary value problem after a sufficiently long time interval. It is assumed that these oscillations influence the string sufficiently long time. The right end $x = L$ remains attached to the spring.

Problem 2. Let $u(x,t)$ be the solution to the initial value problem

$$u_t(x,t) = \int_0^x u(y,t)dy - xu(x,t), \quad 0 \le x < \infty,\ t > 0,$$

with the initial condition $u(x,0) = xe^{1-2x}$. Show that

$$u(x,t) \le \frac{1}{2}, \quad 0 \le x < \infty,\ t \ge 0.$$

Problem 3. Consider equation

$$u_t + au_x = -2u^{3/2},\ -\infty < x < +\infty,\ t > 0,$$

with initial condition $u(x,0) = u_0(x) > 0$.

(a) Solve this initial value problem.

(b) Let for all x, $-\infty < x < +\infty$, $0.3 \le u_0(x) \le 3$. Show that $u(x,t) \to 0$ as $t \to \infty$ in the space $C_B(-\infty,+\infty)$ with the standard norm

$$||u(.,t)|| = \sup_{-\infty < x < +\infty} |u(x,t)|.$$