General Examination: Ordinary Differential Equations (ODEs)

Problem 1. For the equation
$$\frac{d\theta}{dt} = \cos(2\theta) - \cos\theta$$
regarded as a vector field on the circle, find its fixed points and classify them. Then plot the vector field on the circle.

Problem 2. Consider the system
$$\frac{dx}{dt} = rx + ax^2 - x^4,$$
where a and r are parameters with $a, r \in (-\infty, \infty)$.

(i) For each a, there is a bifurcation diagram of x^* vs r. As a varies, the bifurcation diagram may undergo qualitative changes. Plot all the qualitatively different bifurcation diagrams that can be obtained by varying a.

(ii) Summarize your results by plotting the stability diagram in the ra-plane, *i.e.* identify the regions with qualitatively different vector fields and the boundaries of these regions in the (r, a) plane, and then classify the types of bifurcations in these regions and boundaries.

Problem 3. Consider the system
$$\frac{dx}{dt} = 5 - x^2 - y^2,$$
$$\frac{dy}{dt} = 1 - x - y.$$

(a) Find the fixed points and use linear analysis to classify them.

(b) Plot the phase portrait of each linearized system obtained in (a).

(c) Plot the nullclines and the vector field of the system.

(d) Plot the phase portrait of the system.

(e) Identify the stable and unstable manifolds in the phase portraits in (b) and (d) if they exist. What is the significance of recognizing the stable and unstable manifolds in a phase portrait?