General Examination: Real Analysis

Problem 1. Denote $\mathrm{Irr}^2 = \{(x,y) \in \mathbb{R}^2 \mid x,y \notin \mathbb{Q}\}$. Is there a subset $A \subseteq \mathbb{R}^2$ that is not Lebesgue measurable, but such that $B = A \cap \mathrm{Irr}^2$ is measurable?

Problem 2. Determine if the following is true or false.
Let $f \geq 0$ be a bounded measurable function on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} f\,dm = \inf\left\{ \int_{\mathbb{R}} \varphi\,dm \,\middle|\, \varphi \text{ simple, } f \leq \varphi \right\}.$$

Problem 3. Find a non-negative sequence (α_n) in $\mathbb{R}$ s.t.

$$\lim_{n\to\infty} \int_0^{\pi/2} \alpha_n \sqrt{\sin \tfrac{x}{n}}\,dx$$

is finite and nonzero.

Problem 4. Let the operator $A : C[0,2] \to C[0,2]$ be defined by

$$Ax(t) = t + \int_0^t \frac{x(u)}{2+x(u)^2}\,du.$$

Show that A has a unique fixed point in $C[0,2]$.

Problem 5. Let $x_n \in C[0,1]$ and $x_n(0) = 0$ for all $n = 1,2,\ldots$. Suppose that for all $t_1, t_2 \in [0,1]$ and all $n = 1,2,\ldots$ we have

$$|x_n(t_1) - x_n(t_2)| \leq |t_1 - t_2|^{\left(1+\frac{1}{n}\right)^{-n}}.$$

Show that the family $\{x_n\}$ has a limit point in $C[0,1]$.

Problem 6. Let operator $A : \ell^2 \to c$, where c denotes the space of all sequences of real numbers, be defined on $x = (x_1, x_2, \ldots)$ by $A(x) = (y_1, y_2, \ldots)$, where

$$y_n = \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} x_i.$$

(a) Show that $A(x) \in \ell^2$.

(b) Show that norm of A as operator $\ell^2 \mapsto \ell^2$ is less than $\sqrt{2}$.