General Examination

Probability and Mathematical Statistics

Problem 1. For a rolling die, assume the probability of each possible outcome is proportional to the corresponding number of points X.

(a) (5pts) Formulate the probability space for this random experiment.

(b) (5pts) Assume an insider always revealing the parity of X. Determine the sub-σ-field $\mathscr{G}$ due to the parity.

(c) (5pts) Determine the probability distribution $\mathrm{P}(X \leq 3 \mid \mathscr{G})$.

(d) (5pts) Verify the total probability

$$\mathrm{E}[\mathrm{P}(X \leq 3 \mid \mathscr{G})] = \mathrm{P}(X \leq 3).$$

Problem 2. Suppose that X_i's are i.i.d. with $X_i \sim \mathcal{N}(\mu, 1)$, $i = 1, 2, \cdots$.

(a) (5pts) Evaluate the probability density of X_1^2 and then determine the distribution of $\sum_{i=1}^n X_i^2$ for $n \geq 2$.

(b) (5pts) Show that $\bar{X}$ is sufficient to μ.

(c) (8pts) For $\mu = 0$, show that

$$\frac{\sqrt{n}\bar{X}}{\sqrt{(n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2}}$$

is of Student t distribution with degree of freedom $n - 1$.

(d) (7pts) Show that the distribution of $\sum_{i=1}^n (X_i - \mu)^2$ can be approximated by a normal distribution as n is large.

Problem 3. For a simple and random sample $X_1, \cdots, X_n$ from one population with variance σ^2, let

$$S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2.$$

(a) (8pts) Show that S^2 is not unbiased to σ^2.

(b) (8pts) Show that $S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ is a consistent estimator of σ^2.

(c) (8pts) Suppose the population is normal. Show that the modified estimator

$$\frac{n}{n-1}S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

gets the mean squared error $\frac{2\sigma^4}{n-1}$.

Problem 4. Suppose $X_n \xrightarrow{L} X \sim \mathcal{N}(\mu, \sigma^2)$ and independently $Y_n \xrightarrow{L} a \neq 0$ as $n \to \infty$.

(a) (8pts) Determine the limit distribution of $X_n + Y_n$ as $n \to \infty$.

(b) (8pts) Determine the limit distribution of $Y_n X_n^2$ as $n \to \infty$.

Problem 5. (15pts) For a real sequence $b, a_1, a_2, \cdots$ such that $a_n(X_n - b) \xrightarrow{L} Z$ and a real function g continuously differentiable at b, show that

$$a_n[g(X_n) - g(b)] \xrightarrow{L} g'(b)Z \qquad \text{as } n \to \infty.$$