General Examination: Numerical Analysis

Problem 1. Let A be a real $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$.

(a) Define the three basic elementary row operations on the rows of A.

(b) Suppose that A can be transformed into A' by a sequence of p elementary row operations so that $A' = e_p e_{p-1} \ldots e_1(A)$, i.e., A is row equivalent to A' (written $A \approx A'$). Show that $A \approx A'$ if and only if there is a non-singular $m \times m$ matrix C such that $A' = CA$.

(c) If A is an $m \times n$ matrix of rank r, describe in detail a reduced row echelon form (RREF) for A.

(d) Show that if A is $n \times n$, then A is non-singular if and only if $A \approx I_n$, where I_n is the $n \times n$ identity matrix.

Problem 2. Let A be a real non-singular $n \times n$ matrix.

(a) Show that A can be expressed as $A = QR$ where Q is $n \times n$ with orthonormal columns (relative to the Euclidean inner product) and R is non-singular and upper triangular.

(b) Show that Q in part (a) is an **orthogonal matrix**, i.e., $Q^{-1} = Q^t$ (transpose) and that $\|QX\| = \|X\|$ for any n-dimensional column vector X.

(c) Use $A = QR$ to solve the linear system $AX = Y$ for X, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 \\ \frac{5}{6} \\ 3 \end{pmatrix}.$$

Problem 3. Use Newton iteration to solve $f(x) = x^2 - q$ for x, where $q > 0$. Show that if x_n has k correct digits after the decimal point, then x_{n+1} will have at least $2k - 1$ correct digits after the decimal point, provided that $q > 0.006$ and $k \geq 1$.

Problem 4.

(a) Determine all the values of a, b, c, d, e for which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & x \in (-\infty, 1] \\ c(x-2)^2, & x \in [1, 3] \\ d(x-2)^2 + e(x-3)^3, & x \in [3, \infty) \end{cases}$$

(b) Determine the values of the parameters so that the cubic spline interpolates this table

x	0	1	4
y	26	7	25