General Examination: Real Analysis

Problem 1. Let E be a measurable subset of $[0, 1]$. Suppose there exists $\alpha \in (0, 1)$ such that $m(E \cap J) \geq \alpha \cdot m(J)$ for all subintervals J of $[0, 1]$. Prove that $m(E) = 1$.

Problem 2.

(a) State Baire Category Theorem.

(b) Show that the linear space of all polynomials in one variable with real coefficients is not a Banach space in any norm.

Problem 3. Let $f \in L^1(X, \mu)$. Prove that for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left| \int_A f d\mu \right| < \varepsilon$$

whenever A is a measurable subset of X with $\mu(A) < \delta$.

Problem 4. Let K be a compact metric space. Show that if $f : K \to \mathbb{R}$ is a continuous function then f is uniformly continuous.

Problem 5. Prove that the equation

$$f(x) = \int_0^1 e^{-tx} \cos(\alpha f(t)) dt, \quad 0 \leq x \leq 1,$$

has a unique continuous solution if $|\alpha| < 1$.

Problem 6. Let L be the linear subspace of $C[0, 1]$ given by

$$L = \{x(t) \in C[0, 1] : x(t) = mt\}.$$

Let $\psi \in L^*$, $\psi(x) = m$. Let $L_1 = \text{Span}(L, t^2)$. Find a functional $\varphi \in L_1^*$ such that $\varphi|_L = \psi$ and $\|\varphi\| = \|\psi\|$.