General Examination: Probability and Stochastic Processes

Problem 1. For a rolling die, assume that the probability of each possible outcome is proportional to X – the corresponding number of points.

(a) (5pts) Formulate the probability space for this random experiment.

(b) (5pts) Assume an insider always revealing the parity of X. Determine the sub-σ-field $\mathscr{G}$ due to the parity and the probability $P(X \leq 3 \mid \mathscr{G})$.

(c) (10pts) Determine the expectation $\mathrm{E}[X \mid \mathscr{G}]$.

(d) (5pts) Verify the duplicate expectation $\mathrm{E}[\mathrm{E}[X \mid \mathscr{G}]] = \mathrm{E}[X]$.

Problem 2. Suppose that X_i's are i.i.d. with $X_i \sim \mathcal{N}(0, 1)$, $i = 1, 2, \cdots$.

(a) (5pts) Evaluate the probability density of X_1^2 and then determine the distribution of $\sum_{i=1}^n X_i^2$ for $n \geq 2$.

(b) (10pts) Show that the distribution of $\sum_{i=1}^n X_i^2$ can be approximated by normal distribution as n is large.

(c) (10pts) Show that the central Student t distribution can be approximated by standard normal distribution as the degree of freedom is large.

Problem 3. During each unit of time, (i) either 0 (with probability $1 - \lambda$) or 1 (with probability λ) customer arrives for service and joins a single line, and (ii) independently of new arrivals, a single service is completed with probability p or continues into the next period with probability $1 - p$. Let X_n be the total number of customers (waiting in line or being in service) at the nth unit of time and assume $0 < \lambda < p < 1$.

(a) (5pts) Show that $\{X_n\}$ is a birth-death chain on $S = \{0, 1, 2, \cdots\}$.

(b) (10pts) Discuss transience, recurrence, positive-recurrence of the chain.

(c) (5pts) Determine the invariant initial distribution π.

(d) (5pts) Evaluate $\mathrm{E}_\pi[X_n]$.

Problem 4. Let $\{N_t,\ t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and $\{Y_n,\ n \geq 1\}$ independent of $\{N_t,\ t \geq 0\}$ be i.i.d. jumps with

$$\mathrm{P}(Y_i = 1) = \mathrm{P}(Y_i = 2) = 1/2, \quad i = 1, 2, \cdots .$$

Denote $X_t = \sum_{n=0}^{N_t} Y_n$ for $t \geq 0$.

(a) (5pts) Evaluate the expectation $\mathrm{E}[X_t]$.

(b) (10pts) Show that the chain $\{X_t,\ t \geq 0\}$ is of Markov property.

(c) (5pts) Determine the infinitesimal transition generator $\boldsymbol{Q}$.

(d) (5pts) Evaluate the distribution of the sojourn time at each state.