General Examination: Real Analysis

Throughout this paper, m denotes Lebesgue measure on $\mathbb{R}$.

Problem 1. Let a nonnegative measurable function $f : [0,1] \to \mathbb{R}$ be such that $\int_E f\, dm \leq m(E)^2$ for every measurable $E \subseteq [0,1]$. Show that $f = 0$ a.e.

Problem 2. Suppose boundary ∂E of a set $E \subseteq R^n$ has Lebesgue measure 0. Is E necessarily measurable?

Problem 3. Let f be a non-negative function on $\mathbb{R}$. Assume that for all $n \geq 1$

$$\int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} f(x) dm \leq 1.$$

Show that f is Lebesgue integrable and $\int_{\mathbb{R}} f dm \leq 1$.

Problem 4. Consider the weakest topologies τ_1 and τ_2 generated on the set $\mathbb{R}$ of real numbers by the bases $B_1 = \{(a,b) \mid a < b \in \mathbb{R}\}$, $B_2 = \{[a,b) \mid a < b \in \mathbb{R}\}$, respectively.
(a) Are these topologies comparable? If yes, which topology is stronger?
(b) Does the sequence $x_n = \dfrac{(-1)^n}{n}$, $n = 1, 2, \ldots$, converge in τ_1? In τ_2?

Problem 5. Let $A : \ C[0,1] \mapsto C[0,1]$, $Au(x) = u(\frac{x}{2}) + \int_0^1 K(x,y)u(y)dy$ where function $K(x,y)$ has continuous first derivatives in both variables. Is operator A compact?

Problem 6. Let $x \in c_0$ where c_0 is the space of converging to zero sequences with norm $||x|| = \sup_n |x_n|$. Let $D(\psi) \subset c_0$ be the domain of the functional

$$\psi(x) = \sum_{k=2}^{\infty} \frac{x_k}{k \ln k}.$$

(a) Is $D(\psi)$ dense in c_0?
(b) Find $||\psi||$ or show that ψ is unbounded.
(c) Is $\operatorname{Ker} \psi$ dense in c_0?