General Examination: Numerical Analysis and Ordinary Differential Equations

Problem 1. Define the IEEE Standard 754 Floating Point representation of a finite precision number stored in a computer for

(a) single precision representation

(b) double precision representation

(c) Let x be the real number $x = 4/3$. Define $\widetilde{x}$ as the IEEE Standard 754 Floating Point single precision machine representation of x using rounding to the nearest. Determine $\widetilde{x}$.

(d) Using the machine representation $\widetilde{x}$ of x found in (c), evaluate the expression $z = 1 - 3(\widetilde{x} - 1)$. Explain any issues or anomalies with this computation.

Problem 2. (*A hand calculator is required.*) Find ALL of the real zeros of the function $g(x) = 2x - (1 + x^2)\tan^{-1} x$. Find the positive zeros with 4 decimal place accuracies. Verify the accuracies.

Problem 3.

(a) Let A be a real $m \times n$ matrix and let $\|X\|$ denote the Euclidean norm on $\mathbb{R}^n$. Define the associated induced matrix norm $\|A\|$.

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Use the definition above to show that to 4 decimal digits $\|A\| \approx 2.9208 > \sqrt{8}$.

Problem 4. Given an $m \times n$ real matrix A, an n-dimensional real column vector X and an m-dimensional real column vector B, the matrix equation $AX = B$ represents a system of m linear equations in n unknowns.

(a) In case $m = n$ and A is non-singular, provide a detailed description of an algorithm for computing the solution X using the decomposition $PA = LU$, where P, L, and U are $n \times n$ matrices, P a permutation matrix, L lower triangular with 1's on the main diagonal and U upper triangular.

(b) How are the entries of L determined?

(c) Under what sets of conditions can the permutation matrix P be taken as the $n \times n$ identity I_n?

(d) Suppose A is $m \times n$ with $m > n$. Discuss a method for determination of vectors X that minimize $\|AX - B\|^2$ (i.e. the "least squares" solutions), where $\|\cdot\|$ denotes the Euclidean norm. Under what condition does a *unique solution* exist?

(e) In part (d) suppose that A has a large condition number $\kappa(A)$. Describe a numerically reliable technique for determining the unique least squares solution.

Problem 5. Given the equation $\dfrac{dx}{dt} = x \sin x$,

(a) Find all fixed points.

(b) Plot its vector field.

(c) Classify all the fixed points.

(d) Plot the solutions in the tx-plane satisfying the initial conditions $x(0) = \pi/4$, $x(0) = \pi$, $x(0) = 8\pi/7$ and $x(0) = -\pi/6$.

Problem 6. For the differential equation $\dfrac{dy}{dt} = y(2y^2 - y^4 + \alpha)$, where α is a parameter,

(a) Find all bifurcation values of α as α varies on the real line $-\infty < \alpha < \infty$.

(b) Classify the bifurcations obtained in (a).

(c) Plot the bifurcation diagram that also displays the stability of fixed points.

Problem 7. (a) Suppose that the system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

is conserved with the conserved quantity $E(x, y)$. Show that the system has neither spirals, nor stable and unstable fixed points. As a consequence, show that if its linearized system about a fixed point (x_0, y_0) has a center, then (x_0, y_0) is a center as well.

(b) Show that a Hamiltonian system

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y}, \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} \end{aligned}$$

is conserved.

Problem 8. For the system

$$\frac{dx}{dt} = x(2 - x - y),$$
$$\frac{dy}{dt} = y(y - x^2),$$

(a) find fixed points, and use linear analysis to classify them if the analysis applies;

(b) in the phase plane, plot nullclines, and plot the vector field on the nullclines;

(c) sketch the phase portrait using the results of (a) and (b).

Problem 9. (**Bonus problem**) Suppose that $N = N(t)$ is the population density of a species, satisfying the differential equation

$$\frac{dN}{dt} = \alpha - \beta N - \gamma e^{-\delta N},$$

where α, β, γ and δ are positive parameters.

(a) Use dimension analysis to rewrite the equation in the dimensionless form

$$\frac{dx}{d\tau} = a - bx - e^{-x}$$

with the dimensionless parameters $a > 0$ and $b > 0$.

(b) Show that the system in (a) could have no fixed point, one, or two fixed points, depending on the values of a and b. Then classify the fixed points.

(c) Show that bifurcations occur at $a = 1$ and $a = b - b \ln b$. Then classify the bifurcations.

(d) Plot a stability diagram in the ab-plane and identify the regions in the first quadrant where the system has no, one, or two fixed points.