# General Examination: Complex Analysis and Partial Differential Equations

**Problem 1.** Prove Morera's theorem:
Suppose $f$ is a continuous function on a simply connected domain $G$ and

$$\int_\gamma f(z)dz = 0$$

for every simple closed contour $\gamma$ in $G$. Then $f$ is analytic in $G$.

**Problem 2.** Find all entire functions $f$ such that $|f'(z)| \leq \frac{1}{2}|f(z)|$ for all $z \in \mathbb{C}$.

**Problem 3.** Let $f$ be an analytic function in $|z| < 2$, and let $f$ only have zeros at the distinct points $a_1, a_2, \ldots, a_k$, of orders $m_1, \ldots, m_k$, respectively. Further, suppose $|a_j| < 1$ for each $j = 1, \ldots, k$. If $g$ is a given function analytic in $|z| < 2$, find

$$\int_C \frac{f'(z)g(z)}{f(z)} dz,$$

where $C$ is a circle of radius 1 centered at 0.

**Problem 4.** Consider an oscillating string of length $L$ with fixed left end: $u(0,t) = 0$ and fixed right end: $u(L,t) = h = \text{const}$, where $u(x,t)$ is the deviation of the string from horizontal $x$-axis. The linear density of the string is $\rho = \text{const}$ and the tension is $T = \text{const}$. The string undergoes gravitation with constant acceleration $g$ and the resistance force proportional to the deviation. Then, assuming that the $u$-axis is directed upward, the problem is

$$u_{tt} = c^2 u_{xx} - pu - g, \quad c^2 = \frac{T}{\rho}.$$

(a) Find the equilibrium position of the string as $t \to \infty$.
(b) Assume that the string is initially placed in its equilibrium position found above. Assume that at time $t = 0$ the moment of motion $M$ is applied at point $x = x_0$, i.e., the initial speed is $u_t(x,0) = \dfrac{M}{\rho}\delta(x - x_0)$. Solve the problem for $u(x,t)$.

**Problem 5.** Let $u(x,t)$ be the solution to the problem

$$u_t = 3u_{xx} - 4u_x - 2u, \quad 0 < x < 1,$$

with boundary conditions $u(0,t) = 4e^{-t}$, $u(1,t) = \sin(\pi t)$ and the initial condition $u(x,0) = 4 - 4x$. Let $a \le u(x,t) \le b$ for all $0 \le x \le 1$ and $t \ge 0$. Find the best estimates for the bounds $a$ and $b$, which may be $t$-dependent.

**Problem 6.** Consider equation

$$u_t + a(u)u_x = bu + cu^2 + px, \quad -\infty < x < +\infty, \quad t > 0,$$

with initial condition $u(x,0) = u_0(x)$.
(a) Assuming that $c = 0$ and $a, b, p$ are constants, solve the above initial value problem for the corresponding linear equation.
(b) Assume $a(u) = u$ and $b = c = p = 0$. Solve the initial value problem for the corresponding quasilinear equation if

$$u_0(x) = 1 \text{ for } x < 0 \text{ and } u_0(x) = 2 \text{ for } x \ge 0.$$

(c) Let $b = p = 0$ and $a, c = \text{const}$, solve the initial value problem for the corresponding semilinear equation.