General Examination: Real Analysis

Problem 1. Let K be a compact metric space. Show that every isometry $K \to K$ is surjective.

Problem 2. Let A be a non-Lebesgue measurable subset of $[0,1]$. Prove that there exists some $0 < \varepsilon < 1$ such that for any Lebesgue measurable subset E of $[0,1]$ with $m(E) \geq \varepsilon$, the set $A \cap E$ is not Lebesgue measurable. (m denotes the Lebesgue measure on $\mathbb{R}$.)

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone and $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove that the composition $f \circ g$ is Lebesgue measurable.

Problem 4. For a nonnegative Lebesgue integrable function f on $[0,1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} \sqrt[n]{f}\, dm = m(\{x : f(x) > 0\}),$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

Problem 5. Show that a closed proper vector subspace of a normed vector space is nowhere dense.

Problem 6. On the vector space $C[0,1]$ of continuous functions $[0,1] \to \mathbb{R}$ consider the two norms

$$\|f\|_C = \sup_{0\leq x\leq 1} |f(x)| \quad \text{and} \quad \|f\|_1 = \int_0^1 |f(x)| dx.$$

Consider the identity operator $I : (C[0,1], \|\cdot\|_C) \mapsto (C[0,1], \|\cdot\|_1)$.
(a) Is I continuous?
(b) Is I surjective?
(c) Is I open?