General Examination: Algebra and Linear Algebra

Problem 1.

1. Let G be a finitely generated group with the property that each conjugacy class is finite. Show that the center $Z(G)$ has finite index in G.
2. Show that if a group has a proper subgroup of finite index, then it must have a proper normal subgroup of finite index.
3. Give an example of an abelian group that has no proper subgroup of finite index.
4. Give an example of a non-abelian group that has no proper subgroup of finite index.

Problem 2.

1. Show that there is no simple group of order 36.
2. Show that there is no simple group of order p^2q^2, where p,q are distinct primes.

Problem 3. Let R be a ring with 1 such that $x^2 = x$ for all $x \in R$.

1. Show that $2x = 0$ for all $x \in R$.
2. Show that R is commutative.
3. Show that every finitely generated ideal in R is principal.

Problem 4. Let F be the field of fractions of a unique factorization domain D.

1. Let $b \in F$ be a root of a monic $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in D[x]$. Prove that $b \in D$.
2. Prove that $g(x,y) = y^2 - x(x-1)(x+1)$ is irreducible in $\mathbb{Q}[x,y]$.