## General Examination: Real Analysis

# 1. Let $C$ be a non-empty closed subset of a metric space $(X,d)$. Show that the function $f(x) = \inf\{d(x,y), y \in C\}$ is continuous and moreover, $f(x) = 0$ if and only if $x \in C$.

# 2. Give an example of a normed vector space $X$ and a proper subspace $S$ of it, which is isometric to the space $X$.

# 3. Let $X$ be a complete metrtic space and let $x_0 \in X$, $r > 0$. Let $B$ be the closed ball of radius $r$ centered at $x_0$: $B = B_r(x_0) = \{x \in X : \rho(x,x_0) \leq r\}$.
Let the mapping $T$ be Lipshitz-continuous with a Lipschitz constant $c$:

$$\rho(Tx,Ty) \leq c\rho(x,y).$$

Suppose $cr + \rho(Tx_0,x_0) \leq r$. Prove that $T(B) \subset B$ and that mapping $T : B \mapsto X$ has a fixed point.

# 4. Let $f_n(x) = \sum_{k=1}^{n} \left(x^k - x^{2k}\right)$, $x \in E = [0,1]$, and let $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in E$.

(a) Is the function $f$ continuous in $E$? If no, find the Lebesgue measure of the subset in $E$, on which the function $f$ is continuous.

(b) Does the sequence $f_n$ converge to $f$ uniformly in $E$?

# 5. Let the set $E \subset \mathbb{R}$ have a finite Lebesgues measure $m(E) < \infty$. Let $f_n : E \mapsto \mathbb{R}$ and for every $\varepsilon > 0$ $\exists\delta > 0$ such that for all $n \geq 1$ and any measurable set $A \subset E$ with $m(A) < \delta$,

$$\int_A |f_n|\, dm < \varepsilon.$$

Let $f_n \to f$ pointwise a.e. on $E$. Prove that $f$ is integrable over $E$.