General Examination: Complex Variables

Problem 1. Prove or disprove (provide arguments, or give a counterexample):

(a) Let z_1, z_2, z_3, z_4 be nonzero complex numbers. Then at least one of the numbers z_i/z_j ($i, j \in \{1,2,3,4\}$, $i \neq j$) has nonnegative real and imaginary parts.

(b) Suppose the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, $0 < R < \infty$.

Then the Taylor series $\sum_{n=0}^{\infty} a_{2n} z^n$ has a radius of convergence R^2.

(c) Let $f(z) = u(x,y) + iv(x,y)$ be a complex differentiable function on an open connected set G, where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, and $z = x + iy$. Then $f(z) = v_y(x,y) + iv_x(x,y)$ is a complex differentiable function on G. (Here v_x, v_y denote the partial derivatives of v.)

Problem 2. Suppose f is an analytic function on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$, the punctured unit disk. Prove that if $\operatorname{Re} f$ is a bounded function, then f has a removable singularity at 0.

Problem 3. Suppose $f(z)$ is a complex differentiable function on an open set D.

(a) Prove that f has a Taylor series expansion about each point $z_0 \in D$ with a positive radius of convergence $R > 0$.

(b) Assuming additionally that $R < \infty$, show that f has at least one singularity on the boundary of the disk $|z - z_0| < R$.

Problem 4. Let f be an entire function with the property that

$$f(z+m+ni) = f(z) \quad \text{for all } z \in \mathbb{C},\ m,n \in \mathbb{Z}.$$

Prove that f is a constant.

Problem 5. Use complex analysis techniques to evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^7}$.