General Examination: Algebra

1. Let $G = \mathbf{GL}_n(\mathbb{R})$ be the group of nonsingular $n \times n$ matrices over the real numbers $\mathbb{R}$, and let $S = \mathbf{SL}_n(\mathbb{R})$ be the subgroup of G consisting of matrices of determinant 1.

   (i) Show that the subgroup S is normal in G.

   (ii) Show that the quotient G/S is isomorphic to the multiplicative group of $\mathbb{R}$.

2. Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Prove that a Sylow 11-subgroup is contained in the center of G.

3. Let $R = C[0,1]$ be the ring of all continuous real-valued functions on $[0,1]$ with addition and multiplication defined by:

$$(f+g)(x) = f(x) + g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x).$$

   (i) Prove that for any $x_0 \in [0,1]$, $I = \{f \in R \mid f(x_0) = 0\}$ is an ideal in R.

   (ii) Prove that every ideal $I = \{f \in R \mid f(x_0) = 0\}$ is maximal.

   (iii) Prove that there are no other maximal ideals.

4. Let E be an extension field of F, and let $\alpha \in E$ be transcendental over F. Show that if $\beta \in E$ is algebraic over $F(\alpha)$, then there is a nonzero polynomial $f(x,y) \in F[x,y]$ such that $f(\alpha,\beta) = 0$.