General Examination: Real Analysis

Problem 1. Which of the following properties of metric spaces are preserved under any surjective continuous mappings (provide reasons in each case):

(a) completeness,

(b) total boundedness,

(c) separability,

(d) compactness?

Problem 2. Let A be a linear operator $A : l^2 \mapsto l^2$ given by

$$Ax = \Big(\big(1+\tfrac{1}{2}\big)x_1,\ \big(1+\tfrac{1}{2}+\tfrac{1}{4}\big)x_2,\ \dots\ ,\ \big(1+\tfrac{1}{2}+\tfrac{1}{4}+\dots+\tfrac{1}{2^n}\big)x_n,\ \dots \Big)$$

for $x = (x_1, x_2, \dots)$. Is A bounded? If yes, evaluate its norm.

Problem 3. Let $f \in L^2(\mathbb{R})$ and define

$$g(x) = \int_x^{x+1} f(t)dt.$$

Prove that $\lim_{x\to\infty} g(x) = 0$.

Problem 4. Let f be a continuous function on $\mathbb{R}$. Show that the inverse image with respect to f of a Borel set is Borel.

Problem 5. Let (f_n) be a sequence of differentiable functions on $[a,b]$, $-\infty < a < b < \infty$, whose derivatives are uniformly bounded and there is a point $x_0 \in [a,b]$ such that $(f_n(x_0))$ is bounded in $\mathbb{R}$. Prove that (f_n) has a uniformly convergent subsequence in $C[a,b]$.

Problem 6. Determine the limit

$$\lim_{n\to\infty} \int_0^1 \frac{n\cos^2 x}{1+n^2x^2} dx.$$