General Examination: Probability and Stochastic Processes

Name:

- Complete all questions in 3 hours, and make sure your work is clear and readable.
- Base your work on theories and methods covered in lectures, and provide the theoretical basis for key steps.

1. Suppose $X_n \xrightarrow{L} X$ and independently $Y_n \xrightarrow{P} a \neq 0$ as $n \to \infty$.

   (i) (10pts) Prove that $X_n/Y_n \xrightarrow{L} X/a$ as $n \to \infty$.

   (ii) (10pts) Prove that the central Student t distribution converges to the standard normal distribution as the degree of freedom goes to infinite.

2. For a rolling die assume the probability of each possible outcome is proportional to the corresponding number of points.

(i) (10pts) Describe the probability space for the random number of points.

(ii) (10pts) Assume that an insider always tells you whether total number of points is even or not. Determine the conditional probability distribution with respect to the insider information.

(iii) (10pts) Compute the corresponding conditional expectation.

3. During each unit of time, (i) either 0 (with probability $1-\lambda$) or 1 (with probability λ) customer arrives for service and joins a single line, and (ii) independently of new arrivals, a single service is completed with probability p or continues into the next period with probability $1-p$. Let X_n be the total number of customers (waiting in line or being serviced) at the nth unit of time.

(i) (10pts) Show that $\{X_n\}$ is a birth-death chain on $S=\{0,1,2,\cdots\}$.

(ii) (10pts) Discuss transience, recurrence, positive-recurrence.

(iii) (10pts) Assume $\lambda < p$. Calculate the invariant initial distribution π and $\mathrm{E}_\pi[X_n]$.

4. Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and $\{Y_n, n \geq 1\}$ independent of $\{N_t, t \geq 0\}$ be a sequence of i.i.d. jumps with $\mathrm{P}(Y_i = 1) = \mathrm{P}(Y_i = 2) = 1/2$. Denote $X_t = \sum_{n=0}^{N_t} Y_n$ for $t \geq 0$.

(i) (10pts) Show that $\{X_t, t \geq 0\}$ is a Markov chain.

(ii) (10pts) Determine the infinitesimal transition generator $\boldsymbol{Q}$.

(iii) (10pts) Evaluate the distribution of the sojourn time at each state and verify the Kolmogorov backward equations.