General Examination: Probability and Statistics

Name:

- Complete all questions in 3 hours, and make sure your work is clear and readable.
- Base your work on theories and methods covered in lectures, and provide the theoretical basis for key steps.

1. For a rolling die assume that the probability of each possible outcome is proportional to the corresponding number of points.

   (i) (10pts) Describe the probability space for the random number of points.

   (ii) (10pts) Assume that an insider always tells you whether total number of points is even or not. Determine the conditional probability distribution with respect to the insider information.

   (iii) (10pts) Compute the corresponding conditional expectation.

2. Let $X_1, \cdots, X_n$ be a simple and random sample from a population with variance σ^2.

   (i) (10pts) Show that
   $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$
   is an unbiased estimator for σ^2;

   (ii) (10pts) Suppose the population is normal. Calculate the variance of the estimator: $\mathbf{E}\left[\left(S^2 - \sigma^2\right)^2\right]$.

3. Let $X_1, \ldots, X_n$ be a random sample from the distribution

$$f(x|\theta) = \frac{3\theta^3}{x^4}\mathbf{1}_{\{x>\theta\}}.$$

Find the method of moments estimator of θ, prove its consistency and obtain its limiting distribution as $n \to \infty$.

4. The random variables $X_1, X_2, \ldots, X_n$ are i.i.d., they take values in a finite set

$$A_m = \{1, 2, \ldots, m\}$$

and are uniformly distributed over A_m. The random variables $X_i's$ could be numbers that you see, e.g., the number of questions in an exam.

(i) Assume that m is not known and you want to estimate it. Write down the estimator obtained (i) by the method of moments and (ii) by maximizing the likelihood.

(ii) If $\hat{m}$ is the MLE, show that $\hat{m}$ is biased and consistent for m, i.e., $E_m(\hat{m}) \neq m$, but $\hat{m} \to m$ if m is kept fixed and $n \to \infty$.

(iii) Suppose m and n both tend to infinity at the same rate, i.e., $m = k \cdot n, n \to \infty$, with k a fixed constant. Show that $(\hat{m} - m)$ converges in distribution.

5. Let $X_1, X_2, \ldots, X_n$ be Bernoulli random variables with the following joint distribution, depending on $\theta = (\theta_1, \theta_{11}, \theta_{10})$;

$$P(X_1 = 1) = \theta_1$$

$$P(X_i = 1 | X_1, \ldots, X_{i-1}) = \begin{cases} \theta_{11} & \text{if } X_{i-1} = 1 \\ \theta_{10} & \text{if } X_{i-1} = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Find a four dimensional sufficient statistic.