General Examination: Real Analysis

Problem 1. The oscillation of a function f over an interval I is defined as $\operatorname{osc}(f,I) = \sup\limits_{x,y\in I} |f(x)-f(y)|$. Then the oscillation at point x is

$$\operatorname{osc}(f,x) = \inf_{r>0} \operatorname{osc}(f,(x-r,x+r)).$$

Prove that f is continuous at x if and only if $\operatorname{osc}(f,x) = 0$.

Problem 2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuous and $g : \mathbb{R} \mapsto \mathbb{R}$ be measurable. Show that $h(x) = f(g(x))$ is measurable.

Problem 3. Let $\mu(X) < \infty$ and f be Lebesgue's integrable on X. Prove that the Lebesgue's integral is equal to the limit of the Lebesgue's integral sums

$$\int_X f(x)d\mu = \lim_{\lambda\to 0} \sum_k \xi_k \cdot \mu\left(\{x \in X : t_k \le f(x) \le t_{k+1}\}\right),$$

where $\lambda = \sup_k |t_k - t_{k+1}|$ is the diameter of partition t_k and points $\xi_k \in [t_k, t_{k+1}]$ are chosen randomly.

Problem 4. Prove or disprove that the space l^∞_{conv} of converging number sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ is separable, i.e., contains a countable dense subset.

Problem 5. Let operator $A : l^p \mapsto l^p$, $p \ge 1$, be defined as $Ax = (a_1x_1, \ldots, a_nx_n, \ldots)$, where sequence $\{a_n\}$ is bounded and $x = (x_1, \ldots, x_n, \ldots) \in l^p$. Prove that the operator A is compact if and only if $\lim\limits_{n\to\infty} a_n = 0$. (Reminder: an operator A is compact if it maps bounded sets into relatively compact sets.)

Problem 6. Let operator A map a compact metric set (X,ρ) into itself and let $\rho\left(A^2x, A^2y\right) < \rho(x,y)$. Then prove or disprove that the operator A has a fixed point $x_0 \in X$, i.e., $Ax_0 = x_0$.