General Examination: Complex Analysis

Problem 1. Suppose that $\{f_n(z)\}$ is a sequence of analytic functions on an open set U and that for any compact subset K of U, $\{f_n(z)\}$ is uniformly convergent to a function $f(z)$ on K. Show that $f(z)$ is an analytic function on U.

Problem 2. Let $f(z)$ be an analytic function on an open disk $D_r(z_0) = \{z : |z - z_0| < r\}$. If $z = z_0$ is a zero of $f(z)$ with a multiplicity of $k > 0$, show that there is a neighborhood V of $w = 0$ such that for each $\alpha \in V$, $f(z) - \alpha$ has a zero in $D_r(z_0)$ with the same multiplicity k.

Problem 3. $f(z)$ is called a holomorphic function on an open set U if it is differentiable at every point in U, whereas $f(z)$ is an analytic function on U if it has a power series expansion with a positive radius of convergence about each point in U. Show that the above two definitions are equivalent.

Problem 4. Suppose that $f(z) = u + iv$ is an entire function and its real part u is a bounded function on $\mathbb{C}$. Show that $f(z)$ is a constant, i.e. $f(z) \equiv c$ for some constant c and any $z \in \mathbb{C}$.

Problem 5. Use techniques of complex analysis to evaluate

$$\int_0^\infty \frac{dt}{(t^2+1)\cosh(\pi t)}.$$