General Examination: Graph Theory and Combinatorics

Choose any four of the following five questions.

Problem 1. Let $G = (V, E)$ be a graph with node set V and edge set E.

(a) Define the following:

- the minimum degree δ of G;
- the node connectivity κ of G;
- the edge connectivity λ of G.

(b) State the inequality relationship between these three parameters and prove your result.

Problem 2. Let $G = (V, E)$ be a graph with node set V and edge set E.

(a) Define: G is *hamiltonian*.

(b) State and Prove Ore's Theorem regarding hamiltonicity of G.

(c) State Chvátal's Theorem regarding forcible hamiltonicity of a degree sequence, being sure to define *forcible*.

(d) Give an example of a degree sequence (realizable) that does not satisfy Chvátal's condition but for which there exists a realization that is hamiltonian. Give the realization, as well.

Problem 3. Derive a formula for $\sum_{k=1}^{n} k^3$, being sure to explain your approach.

Problem 4.

(a) State and prove Menger's Theorem (regarding the connectivity parameter κ_{st}).

(b) State Hall's Marriage Theorem and prove it using Menger's Theorem.

(c) State the "transversal" version of Hall's Marriage Theorem, being sure to establish the connection.

Problem 5. Let G be a graph and $\vec{G}$ an arbitrary orientation of G.

(a) Define the node-arc incidence matrix S of $\vec{G}$.

(b) Define the Laplacian $H(G)$.

(c) Prove $H(G) = SS^T$.

(d) Prove $H(G)$ is positive semi-definite but *NOT* positive definite