General Examination: Algebra and Linear Algebra

Problem 1. Let G be an abelian group, $K = \{g \in G \mid g^2 = 1\}$, and $H = \{g^2 \mid g \in G\}$ Prove that $G/K \simeq H$.

Problem 2. Let p, q be distinct primes and $n \in \mathbb{N}$ satisfying $q \nmid p^i - 1$ for every $1 \leq i \leq n-1$. Prove that every group G of order $p^n q$ is solvable.

Problem 3. Let R be a finite commutative ring with unity. Prove that every prime ideal in R is maximal.

Problem 4. Let R be a commutative ring with unity, I, J ideals in R.

- Show that $I \cap J$ is an ideal in R.
- Show that $I + J = \{a + b \mid a \in I,\ b \in J\}$ is an ideal in R.
- Show that if $I + J = R$, then $R/(I \cap J) \simeq R/I \times R/J$.