General Examination: Probability and Stochastic Processes

Problem 1. Let X be a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$. Show that X is independent of itself if and only if X is $\mathbf{P}$-a.s. a constant.

Problem 2. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} 9e^{-3x-3y} & x > 0,\ y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbf{E}[X+Y]$.

Problem 3. Prove or give a counterexample that

$$X_n \to X \quad \text{in probability} \quad \Longrightarrow \quad \mathbf{E}[X_n] \to \mathbf{E}[X].$$

Problem 4. Consider a stationary Markov Chain $X_1, X_2, \ldots$ with transition probability $\{p_{ij}\}$, that is,

$$\mathbf{P}\left(X_{n+1} = j \mid X_n = i\right) = p_{ij}, \quad \forall n.$$

Show that if the process has an equilibrium distribution $\{\pi_i\}$, then the reverse Markov chain X_n, X_{n-1}, X_{n-2}, $\ldots$ also has the same equilibrium distribution $\{\pi_i\}$.

Problem 5. Let $M_n = \sum_{i=1}^{n} X_i$ be a simple random walk with

$$\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = \frac{1}{2}, \quad \forall i.$$

Let

$$\tau = \min\left\{n \in \mathbb{N} : M_n = 10 \text{ or } M_n = -5\right\}.$$

Show that

$$\mathbf{P}(M_\tau = 10) = \frac{1}{3}.$$