General Examination: Probability and Statistics

Problem 1. Let X be a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$. Show that X is independent of itself if and only if X is $\mathbf{P}$-a.s. a constant.

Problem 2. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} 9e^{-3x-3y} & x > 0,\ y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbf{E}[X+Y]$.

Problem 3. Prove or give a counterexample that

$$X_n \to X \quad \text{in probability} \quad \Longrightarrow \quad \mathbf{E}[X_n] \to \mathbf{E}[X].$$

Problem 4. Let $X_1, \ldots, X_n$ be independent identical Bernoulli(p) random variables.

(a) Find the maximum likelihood estimator (MLE) of the probability p.

(b) Find the maximum likelihood estimator (MLE) of $\pi = \dfrac{p}{1-p}$.

Problem 5. Let $X_1, \ldots, X_n$ be independent identical Poisson(λ) random variables, and let $\bar{X}$ and S^2 be the sample mean and variance, respectively. Define

$$W_a\left(\bar{X}, S^2\right) = a\bar{X} + (1-a)S^2.$$

Give and verify a value of a, so that W_a is a best unbiased estimator of λ.