General Examination: Complex Variables

Problem 1. Prove or disprove (refer to an appropriate theorem or give a counterexample):

(a) For any $z \in \mathbb{C}$, at least one of the numbers z, z^2, z^3 has non-negative real part.

(b) If $f(z)$ is a non-constant entire function, then $f(z) \to \infty$ when $z \to \infty$.

(c) Suppose the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, $0 < R < \infty$. Then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ represented by this Taylor series must be unbounded on the open disc $|z| < R$.

(d) Let $f(z) = u(x,y) + \mathrm{i}\, v(x,y)$ be an analytic function on a domain G, where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, and $z = x + \mathrm{i}\, y$. Then $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y}$ at all points of G.

Problem 2. State and prove Maximum Modulus Principle.

Problem 3. Prove the Fundamental Theorem of Algebra using

(a) Rouché Theorem;

(b) Liouville's Theorem.

(The Fundamental Theorem of Algebra asserts that every non-constant polynomial with complex coefficients has at least one complex root.)

Problem 4. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, use complex analysis techniques to evaluate

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(2bx)\, dx, \quad b > 0.$$