General Examination: Algebra

Problem 1. Let $\mathbb{Q}$ be the additive group of rational numbers and $\mathbb{Z}$ its subgroup of integers. Prove the following:

(a) $\mathbb{Q}/\mathbb{Z}$ contains an element of order n for every $n \in \mathbb{N}$.

(b) $\mathbb{Q}/\mathbb{Z}$ contains a unique subgroup of order n for every $n \in \mathbb{N}$.

(c) Every finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.

Problem 2. Prove the following:

(a) $|\mathbf{Aut}(\mathbb{Z}_7)| = 6$.

(b) Show that a group of order 63 contains an element of order 21.

Problem 3. A **local** ring is a commutative ring with $1 \neq 0$ that has a unique maximal ideal. Show that a ring R is local if and only if the set of non-units in R is an ideal.

Problem 4. Consider $\mathbb{Z}[\frac{1}{2}] = \left\{ \frac{a}{2^n} \mid a, n \in \mathbb{Z},\ n \geq 0 \right\}$ – the smallest subring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\frac{1}{2}$.

(a) Let $\langle 2x-1 \rangle$ be the ideal in $\mathbb{Z}[x]$ generated by the polynomial $2x-1$. Show that $\mathbb{Z}[x]/\langle 2x-1 \rangle \simeq \mathbb{Z}[\frac{1}{2}]$.

(b) Prove that $\langle 2x-1 \rangle$ is not maximal in $\mathbb{Z}[x]$ by finding an ideal I in $\mathbb{Z}[x]$ such that

$$\langle 2x-1 \rangle \subsetneqq I \subsetneqq \mathbb{Z}[x].$$