Doctoral Program | **Department of Mathematical Sciences**
Spring 2013 | Stevens Institute of Technology

General Examination: Real Variables

Problem 1. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set, i.e., it can be represented as a countable intersection of open sets.

Problem 2. Let $E \subset l^1$ be a set of $x = (x_1, x_2, \ldots)$ such that

$$|x_n| \leq \frac{\ln\ln(2+n)}{n(\ln(1+n))^3}, \qquad n = 1, 2, 3, \ldots$$

Let T be a mapping from E into itself such that $||T(x) - T(y)||_{l^1} < ||x - y||_{l^1}$. Is it true that T must have a unique fixed point?

Problem 3. Let E be a bounded Lebesgue measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$, is pairwise disjoint. Show that set E has zero measure.

Problem 4. Let f be Lebesgue integrable over finite interval $[a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f(x)\cos(nx)dx = 0.$$

Problem 5. Let K be a compact subset of a metric space X, and let O be an open set containing K. Show that there is an open set U such that $K \subset U \subset \overline{U} \subset O$, where $\overline{U}$ is the closure of U.

Problem 6. Are there any missing conditions in the following statement?
Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise on a measurable set E to a real-valued function f. Then for each $\varepsilon > 0$ there is a closed set F contained in E for which $f_n \to f$ uniformly on F and $m(E \setminus F) < \varepsilon$.