General Examination: Probability and Stochastic Processes

Problem 1. Let X_1 and X_2 be independent, unit exponential random variables (so the common density is $f(x) = e^{-x}, x > 0$). Define $Y_1 = X_1 - X_2$ and $Y_2 = X_1/(X_1 - X_2)$. Find the joint density of Y_1 and Y_2.

Problem 2. Let $X_1, \ldots, X_{101}$ be 101 independent random variables uniformly distributed on the unit interval $(0,1)$. Let M be the median value of $X_1, \ldots, X_{101}$. Find the numerical probability that $M > 0.52$.

Problem 3. Suppose we can model the proportion of a certain type of bacteria in a culture using a standard Brownian motion. Specifically, B_t denotes this proportion at time t and we assume that $B_0 = 0$. (Recall that a standard Brownian motion is a Gaussian process starting at 0 at time $t = 0$, with independent, identically distributed increments.) Suppose we let the process run and, after 1 hour, we observe $B_1 = 0.5$. That is, half of the solution now contains the bacteria. Given this information find the distribution of the proportion of bacteria after 30 min (i.e., $B_{1/2}$).

Problem 4. A fair six-sided die is rolled repeatedly until all 6 faces appear at least once. Let X denote the number of rolls necessary. Find the mean and standard deviation of X.

Problem 5. Flynn spends his evenings throwing darts at a circular dartboard. The points of impact for different throws are i.i.d. random vectors with common density $f(x,y)$. Here, x and y are the horizontal and vertical coordinates of a point (measured in centimeters) relative to the center of the dartboard. Assume that $f(x,y)$ is continuous and that $f(0,0) > 0$ (i.e., the 2-dimensional density is positive at the board's center). Let D_k be the distance in centimeters from the center of the dartboard to the point of impact of the k^{th} throw. Let $M_n = \min\{D_1, \ldots, D_n\}$ be the distance from the center to the closest hitting point among the first n throws. Find a sequence of constants a_n so that $a_n M_n$ converges in distribution as $n \to \infty$. Prove convergence in distribution, and identify the limiting distribution.