**Doctoral Program** | **Department of Mathematical Sciences**
Spring 2013 | Stevens Institute of Technology

**General Examination: Optimization Theory and Methods**

**Problem 1.** A probability space $\Omega$ has $n$ elementary events occurring with probabilities $p_1$, $p_2$,..., $p_n$, respectively. A random variable $X$ on this space can be identified with a vector $x \in \mathbb{R}^n$, $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$, where each $x_i$, $i = 1, \dots, n$, is the value of $X$ in elementary event $i$. We use $\mathbb{E}[\cdot]$ for the operation of expected value.

**(a)** Consider the function

$$f(x) = \mathbb{E}[X] + \kappa \mathbb{E}\big[\max\big(0, X - \mathbb{E}[X]\big)\big],$$

where $\kappa \geq 0$ is a parameter. Prove that $f(\cdot)$ is convex and positively homogeneous, that is, $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$.

**(b)** Calculate the subdifferential of $f(\cdot)$ at $x = 0$.

**(c)** Calculate the subdifferential of $f(\cdot)$ at $x \neq 0$.

**(d)** Prove that if $\kappa \in [0, 1]$ then every element $g \in \partial f(0)$ satisfies the conditions:

$$g \geq 0, \qquad \sum_{i=1}^{n} g_i = 1,$$

that is, $g$ may be interpreted as a vector of probabilities.

**Problem 2.** Consider the quadratic programming problem

$$\begin{aligned} &\min \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle \\ &\text{s.t. } \langle a_i, x\rangle = b_i, \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$, all matrices and vectors of appropriate dimensions, and with a positive definite $Q$. We assume that $m < n$.

**(a)** Formulate the augmented Lagrangian function for this problem.

**(b)** Suppose that we are using the pre-conditioned conjugate gradient method for minimizing the augmented Lagrangian with the pre-conditioner $V = Q^{-1}$. How many iterations will be sufficient to find the minimum of the augmented Lagrangian?

**(c)** For $Q = I$ and $b = 0$ derive the formula for updating the multipliers by the augmented Lagrangian method.

**(d)** Formulate the dual method for solving problem (1). Discuss the pros and cons of the dual method vs. the augmented Lagrangian method.