General Examination: Complex Analysis

Problem 1. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R > 0$. Show that

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is entire and that for $r \in (0, R)$, there is a constant M such that $|h(z)| \leq M e^{|z|/r}$.

Problem 2. Let $f(z)$ be an analytic function in $\mathbb{C}$, and let $f(z)/z \to 0$ as $|z| \to \infty$. Prove that $f(z)$ is a constant.

Problem 3. Prove or disprove the following statements:

(a) If $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R > 1$, then

$$\frac{1}{2\pi} \int_{|z|=1} |f(z)|^2 \, |dz| = \sum_{n=0}^{\infty} |a_n|^2.$$

Problem 4. Evaluate the following integrals:

(a) $\displaystyle \int_0^{2\pi} e^{e^{i\theta}} d\theta.$

(b) $\displaystyle \int_{|z|=1} \frac{|dz|}{|z-a|^2},$ where $a \in \mathbb{C}$ such that $|a| < 1$.

Problem 5. Use complex analysis techniques to evaluate

$$\int_0^{\infty} \frac{\ln x}{\sqrt{x}\,(x+1)^2} \, dx.$$