General Examination: Real Analysis

Problem 1. Show that if (X,d) is a separable metric space, then its cardinality cannot exceed continuum.

Problem 2. Let $\{f_n\} \subset C[a,b]$ be a sequence of continuous functions converging to f in $C[a,b]$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of $[a,b]$ such that $a_n \to a$, $b_n \to b$. Prove or disprove that

$$\lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)dx = \int_a^b f(x)ds.$$

Problem 3. Show that the Borel σ-algebra on $\mathbb{R}^n$ coincides with the σ-algebra generated by the compact sets.

Problem 4. Let $\{E_n\}$ be a sequence of nonempty Lebesgue measurable subsets of $[0,1]$ such that $\lim\limits_{n\to\infty} \mu_L(E_n) = 1$. Show that $\forall\, \varepsilon \in (0,1)$, $\exists$ a subsequence $\{E_{k_n}\} \subset \{E_n\}$ such that $\mu_L\left(\cap_{n=1}^{\infty} E_{k_n}\right) > \varepsilon$.

Problem 5. Let $f_n(t) = t^{3n} - 2t^{2n} + t^n$. Does the sequence $\{f_n\}$ converge uniformly on $[0,1]$? If no, is there a subset of measure 0.9 on which $\{f_n\}$ converges uniformly?

Problem 6. A series $\sum_{n=1}^{\infty} x_n$ is absolutely summable if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Show that a normed space X is a Banach space if and only if every absolutely summable series in X is convergent.

Problem 7. Prove or disprove the following statement. Let $\{f_n\}$ be a sequence of measurable functions on a set E of finite measure such that $|f_n| \le M$ on E for all $n \in \mathbb{N}$ and some $M > 0$. If $\{f_n\}$ converges to f pointwise, then $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$. If the statement is false, what should be modified to make it true?