General Examination: Complex Analysis

Problem 1. Let D be the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$, and let $f : D \to D$ be bijective and analytic in D. Prove or disprove that if f has at least two fixed points in D then f is the identity function.

Problem 2. Determine the number of zeros, counted with multiplicity, of the equation

$$z^6 - 4z^5 + 9z - 1 = e^z$$

in the annulus $1 \le |z| \le 2$.

Problem 3. Suppose the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 64. What can be asserted about the annulus of convergence of the Laurent series

$$\sum_{n=1}^{\infty} a_{3n+1} z^{-2n} + \sum_{n=0}^{\infty} a_{3n} z^{2n}?$$

Problem 4.

(a) Let f be an analytic function in the open right-half plane ($\operatorname{Re} z > 0$) such that $f(1/n) = 0$ for all $n \in \mathbb{N}$. Prove or give a counterexample that $f(z) \equiv 0$.

(b) Find all entire functions f such that $f'(1/n) = 2f(1/n)$ for all $n \in \mathbb{N}$.

Problem 5. Use complex analysis techniques to evaluate

$$I_n = \int_0^{\infty} \frac{\ln x}{1 + x^n} dx, \qquad n \in \mathbb{N}.$$

What is $\lim_{n\to\infty} I_n$?