General Examination: Algebra

Problem 1. Let H be a subgroup of G with the property: if for any $a, b \in G$ there is $g \in G$ such that $b = g^{-1}ag$, then $b = h^{-1}ah$ for some $h \in H$. Prove that $[G, G] \leq H$.

Problem 2. Let G be a finite group.

(a) Let $H \leq G$. Prove that the number of distinct conjugates of H in G is $[G : N_G(H)]$.

(b) Show that if G has n_p Sylow p-subgroups, then G has a subgroup of index n_p.

Problem 3. Let $D = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Show that

(a) D is a subring of $\mathbb{R}$.

(b) D is a PID.

(c) $\sqrt{3} \notin D$.

Problem 4. Let $f(x) = x^4 + 4x^2 + 2$, and let E be a splitting field of $f(x)$ over $\mathbb{Q}$. Prove that the Galois group of E over $\mathbb{Q}$ is cyclic of order 4.