General Examination: Real Variables

Problem 1. Give examples of
(a) A sequence of Lebesgue integrable functions $\{f_n\}$ such that $f_n \to 0$ a.e. on $[0,1]$, but $\int_0^1 f_n \to \infty$.

(b) A sequence of functions $f_n(x)$ such that $f_n \to 0$ in measure on $x \in [0,1]$, but $\lim\limits_{n\to\infty} f_n(x)$ exists for no $x \in [0,1]$ (i.e., f_n converges nowhere).

Problem 2. Is there a sequence of Lebesque measurable sets $\{E_n\} \subset \mathbb{R}$ such that $E_{n+1} \subset E_n$ for $n \geq 1$ with $\bigcap\limits_{n=1}^{\infty} E_n = [0,1]$ but $m(E_n) = \infty$ for $n \geq 1$?

Problem 3. Prove that the space of converging to zero sequences $\{x_k\}_{k=1}^{\infty}$, $\lim\limits_{k\to\infty} x_k = 0$, with norm $\|x\|_\infty = \sup\limits_{1\leq k<\infty} |x_k|$ is complete.

Problem 4. Let M be a compact metric space and suppose that $f : M \mapsto M$ satisfies $d(f(x), f(y)) < \alpha d(x,y)$ for any $x \neq y$. Find all numbers α for which the mapping f has a unique fixed point.

Problem 5. Let $f(x) = \sum\limits_{k=1}^{\infty} \frac{1}{k} e^{-k(x-k)^2}$ for $x \in \mathbb{R}$. Is $f \in L_1(\mathbb{R})$?

Problem 6. Does $\frac{10}{13}$ belong to the Cantor discontinuum?
(The Cantor discontinuum is a set obtained from $[0,1]$ by taking away the centered open one-thirds of all intervals, i.e. start from $[0,1]$ and take away $\left(\frac{1}{3}, \frac{2}{3}\right)$, then take away $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $(7/9, 8/9)$, and so on.)