General Examination: Complex Analysis

Problem 1.

(a) Find the number of zeros for the polynomial $z^8 - 4z^5 + z^2 - 1 = 0$ inside the unit disk $|z| \leq 1$.

(b) Suppose that f is analytic and has a zero of order m at the point z_0. Show that the function $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\text{Res}[g, z_0] = m$.

Problem 2. Prove or disprove the following assertions:

(a) If $f(z) = u + iv$ is continuous and the Cauchy-Riemann equations are satisfied for $z = 0$, then $f'(0)$ exists.

(b) If $f(z)$ is differentiable at $z = z_0$, then it is analytic at z_0.

(c) If $\sum a_n z^n$ has radius of convergence R, then $\sum n^3 a_n z^n$ has radius of convergence R.

(d) Let $f(z) = u + iv$ be an entire function and $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant. Hint: consider $F(z) = e^{-f^2(z)}$.

Problem 3. Use complex analysis techniques to evaluate the integral

$$I = \int_0^\infty \frac{x^2}{\cosh x} dx.$$