General Examination: Real Variables

Problem 1. Suppose that $f:\ \mathbb{Q} \mapsto \mathbb{R}$. ($\mathbb{Q}$ is the set of rational numbers.)

(a) Let f be Lipshitz. Show that f extends to a continuous function $h:\ \mathbb{R} \mapsto \mathbb{R}$. Is h unique? Note: f is Lipshitz if for all x, y, $\exists C > 0,\ |f(x) - f(y)| \leq C \cdot |x - y|$.

(b) Let f be continuous. Is it possible to form such a unique continuous extension h?

Problem 2. Let d be a metric on a complete space M. Let mapping $f:\ M \mapsto M$ be such that for all $x, y \in M$, $\exists \alpha < 1$, $\beta < 1$, such that

$$\frac{d(f(x), f(y))}{1 + \beta d(f(x), f(y))} < \frac{\alpha d(x, y)}{1 + d(x, y)}.$$

Prove or disprove: Mapping f has a unique fixed point.

Problem 3. Let $\{f_n\}$ be a sequence of Lebesgues measurable functions and let $f:\ X \mapsto \mathbb{R}$. Assume that for every $\varepsilon > 0$ and the Lebesgue's measure μ,

$$\lim_{n\to\infty} \mu(\{x \in X:\ |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Show that there is a subsequence of $\{f_n\}$ that converges to f almost everywhere.

Problem 4. Let $f_n(t) = ne^{-n^2 t}[\sqrt{1 + \ln(1 + nt)} - 1]$, $t \in [0, 1]$. Does this sequence converge uniformly on $[0, 1]$? If no, find a subset of $[0, 1]$ of measure 0.99 on which f_n converges uniformly.

Problem 5. Is it true that $\|A\| = 2$ for linear operator $A:\ C^1[0, 1] \mapsto C[0, 1]$,

$$Ax(t) = x(0) - x(1).$$

Problem 6. Which condition is missing in the following theorem:
Let $\{f_n\}$ be a sequence in L_1 and suppose that $f_n \to f$ pointwise. Then $f \in L_1$ and $\int f_n \to \int f$ as $n \to \infty$ (the integrals are understood in the Lebesgue's sense).