General Examination: Real Variables

Problem 1.

(a) Find a value of α and function $f(x)$, $f : \mathbb{R} \mapsto \mathbb{R}$, such that $[f(x)]^\alpha$ is measurable but $f(x)$ is not measurable.

(b) Find all numbers α for which the measurability of $[f(x)]^\alpha$ implies the measurability of $f(x)$.

Problem 2. Let $f_n(t) = n^2 e^{-nt}(e^t - \cos t - t)$, $t \in [0,1]$. For every $\delta \in (0,1)$, find a set of Lebesgue measure $1-\delta$ on which f_n converges uniformly when $n \to \infty$.

Problem 3. For which values α and β is the function $f(x) = x^\alpha \sin\left(x^\beta\right)$, defined for $x \in (0,1]$,

(a) Lebesgue integrable?

(b) improperly Riemann integrable?

Problem 4. Let $\mathcal{P}$ be the set of all polynomials in $C[a,b]$. Determine whether $\mathcal{P}$ is open, closed, or neither.

Problem 5. Let $\mathcal{B}$ be a Banach space, and let $A : \mathcal{B} \mapsto \mathcal{B}$ and $f : \mathcal{B} \mapsto \mathbb{R}$ be linear operators. Using the Hahn-Banach theorem show that

$$\sup_{\|f\| \le 1, \|x\| \le 1} |f(Ax)| = \|A\|.$$

Problem 6. Determine whether the operator

$$A : \ C[-1,1] \mapsto C[-1,1], \quad Ax(t) = \frac{1}{2}[x(t) + x(-t)]$$

is compact.