General Examination: Part II

Problem 1. Let A, B, and K be minimal normal subgroups of a group G with $K \neq A$, $K \neq B$, and $K \subseteq AB$. Prove that

(a) $KA = AB = KB$

(b) $A \simeq K \simeq B$

(c) AB is abelian

Problem 2. Let

$$f_n(x) = \frac{n(e^x - e)}{1 + n^2 - n^2 \cos(x-1)}, \qquad x \in [0,2].$$

Given $\delta > 0$, find the Egorov set E_δ where the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent.

Problem 3. Consider a Sobolev space $W_2^1[0,2]$ with the inner product

$$\langle x, y\rangle = \int_0^2 [x(t)y(t) + x'(t)y'(t)]dt.$$

Let a functional $F : W_2^1[0,2] \mapsto \mathbb{R}$ be defined by $F(x) = x(1)$. Is it possible to find $y \in W_2^1[0,2]$ such that $F(x) = \langle x, y\rangle$ for all $x \in W_2^1[0,2]$? If "yes," find such $y \in W_2^1[0,2]$.

Problem 4. Using complex analysis techniques, evaluate the integral

$$I_n = \int_0^\infty \frac{\ln x}{1+x^{2n}}\, dx, \qquad n \in \mathbb{N}.$$

What is $\lim\limits_{n\to\infty} I_n$?